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Reflection in quadratic surfaces*

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Abstract

"The transparent cup" is the title of pictures which show an interesting phenomenon: The circular boundary c of the depicted plate appears as an ellipse which seems to coincide with the view of the reflection of c in the coffee-cup. Is this just by chance or is there a geometric theory behind?

In one example the circle c is the focal circle of the reflecting one-sheet hyperboloid, and for this particular case the displayed phenomen is a consequence of focal properties of quadratic surfaces. The tangent cones drawn from a fixed point P to a family of confocal quadrics are confocal and have therefore coinciding axes. These axes are the surface normals to the particular quadrics passing through P. Also the cones connecting P with the focal conics are included in the considered set of confocal cones. Therefore, all focal conics share the property: In each perspective, the images of these curves and their reflections belong to the same conic.

The goal of the paper is to highlight the geometric background, i.e., to focus on confocal conics and their spatial counterparts.

Keywords: confocal conics, confocal quadrics, reflection in quadrics

MSC: 51N20, 51N15, 68U05

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1. Reflection in conics and vertical cylinders

The stimulus for this article is a photograph showing a coffee-cup, which is made of ceramics and stands on a plate[1]. The cup looks transparent since the circular boundary of the plate is completely visible, even its section behind the cup. This apparent transparency is caused by the reflection in the cup: The mirror of the plate's visible boundary appears as an exact continuation of itself. Similar effects can be seen in Figure 1. Is this incidental, or is there a theory behind?

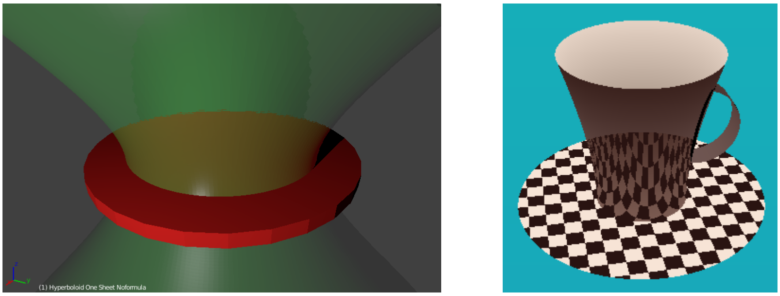

Figure 1: Why does the bounding circle of the plate continue in the reflection? (By courtesy of Kuno Knöbl [4])

Just to fix the terminology, we emphasize that under *'reflection'* in a conic or quadric we understand the physical reflection and not the projective inversion in a quadric[2]. We study the physical reflection in its geometric idealization, which is defined as a transformation applied, in general, to non-directed lines l in the following way: at each point P of intersection with the mirror $\mathcal{R}$, i.e., the reflecting curve or surface, the line l is reflected in the tangent plane τ_P or the normal line n_P to $\mathcal{R}$ at P.[3] The line l can have more than one point of intersection with $\mathcal{R}$ and hence more than one image. Note that each tangent line at P to $\mathcal{R}$ remains fixed.

To begin with, we recall the optical property of conics (see Figure 2, left). The reflection in an ellipse transforms rays emanating from one focus onto rays passing through the other focus. The same holds for hyperbolas when we ignore the orientation of the line. And finally, this optical property is also valid for each parabola when the ideal point of its axis is accepted as the second focus. Since the tangents drawn from a point X to an ellipse share the angle bisectors with the pair of lines connecting X with the focal points [1, p. 42], we can formulate a more general optical property (see Figure 2, right).

[1]See `http://imgur.com/N10ESfl`, retrieved April 2017.

[2]The latter is also known under the name 'projective inversion'; it is a rational transformation where corresponding points are conjugate with respect to ('w.r.t.', in brief) a given quadric and collinear with a given center.

[3]In the two-dimensional case, the reflection in any smooth curve preserves the density $dp \wedge d\varphi$ of oriented lines (satisfying $x\cos\varphi + y\sin\varphi = p$). For further details note [3, p. 6].

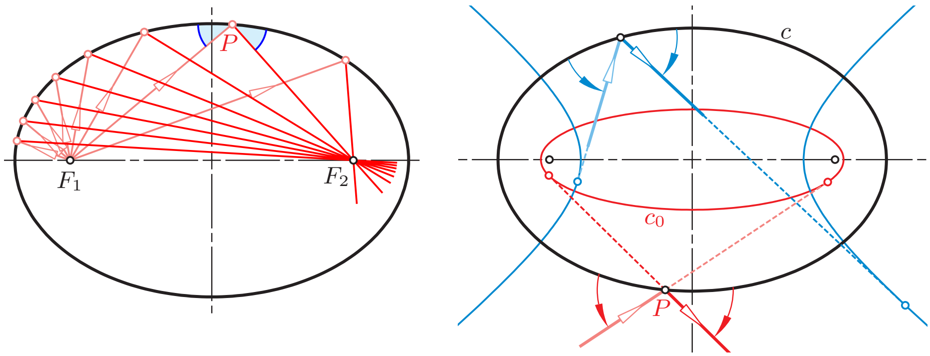

Figure 2: Optical properties of ellipses

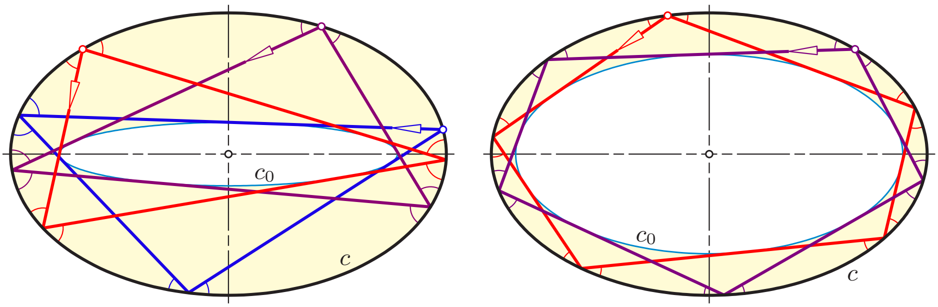

Figure 3: Closed billiards with three or five reflections in an ellipse

Theorem 1.1. *If any ray is reflected in a conic c then the incoming and the outgoing ray are tangent to the same conic c_0 being confocal with c.*

We recall that two conics are called *confocal* if they share the focal points. If in the family of confocal ellipses the minor semi-axis tends to zero the ellipse degenerates into the segment bounded by the two foci. This reveals that the statement of Theorem 1.1 includes the original optical property, too. Analogous degenerations show up as limits of confocal hyperbolas or parabolas.

Iterated reflections of any ray produce *billiards*. Due to Theorem 1.1, billiards in an ellipse c are always circumscribed to another ellipse c_0 being confocal with c. If one billiard inscribed in c and circumscribed to c_0 closes after n reflections then all these billiards close, independently of the choice of the initial point on c (Figure 3). This is a well known example of a Poncelet porism [1, p. 429ff]. All these closed billiards have even the same length, due to Graves' theorem (see [3] or [9] with much more details on billiards and reflections). By the same token, similar properties hold for billiards between two confocal ellipses (Figure 4).

We continue with a rather popular case of a reflection which is often used for producing anamorphoses [5]: Let a right cylinder $\mathcal{R}$ in vertical position be the reflector. As illustrated in Figure 5, if observed from the center C, a point Q of the horizontal ground plane is visible at $P \in \mathcal{R}$. We call P an *reflected image of Q in $\mathcal{R}$ w.r.t. the center C*. The surface normal n_P to the cylinder at P is horizontal. Therefore the two segments PQ and PC of the reflected ray have the same inclination, and n_P is the interior angle bisector of $\angle QPC$, also, when seen in the top view.

As a consequence, for given center C and point Q, a reflected image $P \in \mathcal{R}$ has its top view P' on a *strophoid*, a curve of degree 3 [7]. This is the locus of points X in the ground plane such that a bisector of the angle QXC' passes through a given center M', which in our case coincides with the top view of the axis of $\mathcal{R}$ (Figure 6). Obviously, there is a second point of intersection between the strophoid and the cylinder $\mathcal{R}$ such that the interior angle bisector of $\angle QP'C'$ passes through M'. This shows that point Q can (theoretically) have two reflected images $P, \overline{P} \in \mathcal{R}$; the second one $\overline{P}$ lies on the back wall.

Figure 7 shows also the trajectory q of Q when a reflected image P on $\mathcal{R}$ runs along the horizontal circle $p \subset \mathcal{R}$. These trajectories are circular only in two particular cases: Either $P \in \mathcal{R}$ lies in the ground plane or P has exactly half of the height of C over the ground plane. Otherwise, the trajectories are *Pascal limaçons*.

This can be proved as follows (see Figure 7, left): The reflection at $P \in \mathcal{R}$ acts like the reflection in the surface normal n_P and maps the line PC onto the line PQ. If P has the height z over the ground plane, then the reflection in n_P maps Q onto a point P_2 in the height $2z$ on the line PC . Let P run with angular velocity ω along the parallel circle $p \subset \mathcal{R}$. Then the intersection point P_2 of CP with the plane in the height $2z$ runs with the same angular velocity ω on a horizontal circle p_2 with center M_2 on the cone connecting p with C.

In the top view we obtain Q' when $P_2' \in p_2'$ is reflected in n_P', which rotates with angular velocity ω about M'. This shows that the trajectory q of Q is traced when a first bar $M'M_2'$ rotates about M' with angular velocity 2ω while a second bar

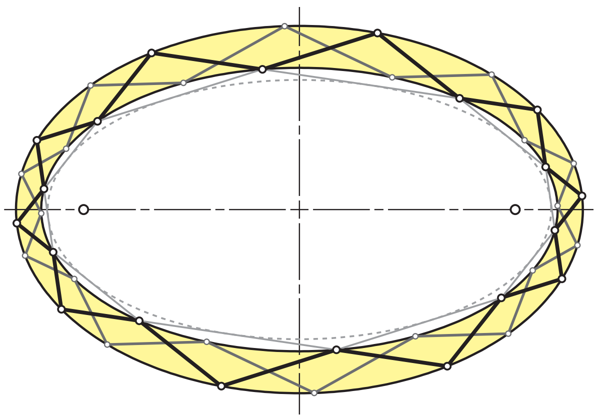

Figure 4: Closed billiards between confocal ellipses (20 reflections)

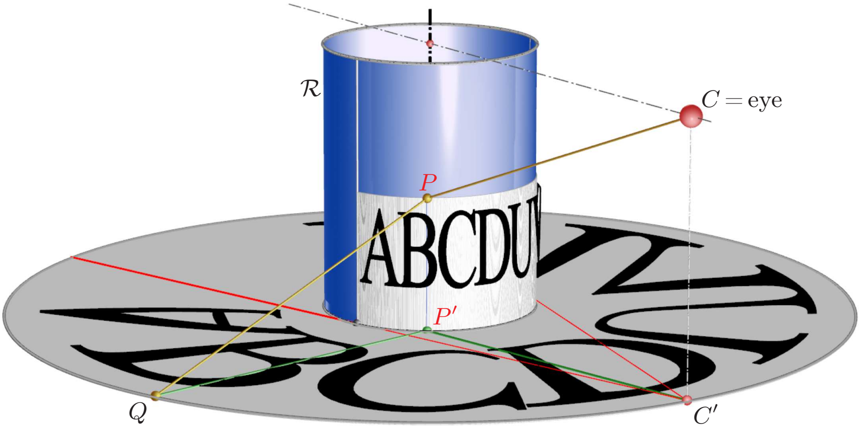

Figure 5: Reflection in a right cylinder $\mathcal{R}$: point Q in the ground plane and a reflected image P (by courtesy of Georg Glaeser)

$M_2'Q_0'$ rotates with the (absolute) velocity ω. A dyad $M'M_2'Q_0'$ moving this way generates as path of its endpoint a particular trochoid, namely a Pascal limaçon q' [10, p. 155], provided that no moving bar has length zero.

2. Confocal quadrics

The word 'quadric' stands now for regular surfaces of degree 2, i.e., for those of full rank 4. Of course, surfaces of degree 2 can also be cylinders or cones (rank 3), pairs of planes (rank 2), or double-counted planes (rank 1). In the projective setting, when cones of degree 2 are regarded as sets of tangent planes, they are dual to conics.

Definition 2.1. Two quadrics are called *confocal* if they have common axes and they intersect each plane of symmetry along confocal conics.

Let $\mathcal{E}$ be a tri-axial ellipsoid with semiaxes a, b and c in standard position. Then the one-parameter set of quadrics being confocal with $\mathcal{E}$ is given as

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=1 \quad \text{for} \quad k\in\mathbb{R}\setminus\{-a^2,-b^2,-c^2\}. \tag{2.1}$$

In the case $a > b > c > 0$ this family includes (see Figure 8)

$$\begin{array}{rll} & -c^2 < k < \infty & \text{tri-axial ellipsoids } \mathcal{E}, \\ \text{for} & -b^2 < k < -c^2 & \text{one-sheet hyperboloids } \mathcal{H}_1, \\ & -a^2 < k < -b^2 & \text{two-sheet hyperboloids } \mathcal{H}_2. \end{array}$$

Their intersections with the plane $z = 0$ share the focal points $(\pm\sqrt{a^2-b^2}, 0, 0)$. In $y = 0$ the common foci are $(\pm\sqrt{a^2-c^2}, 0, 0)$, and in $x = 0$ $(0, \pm\sqrt{b^2-c^2}, 0)$.

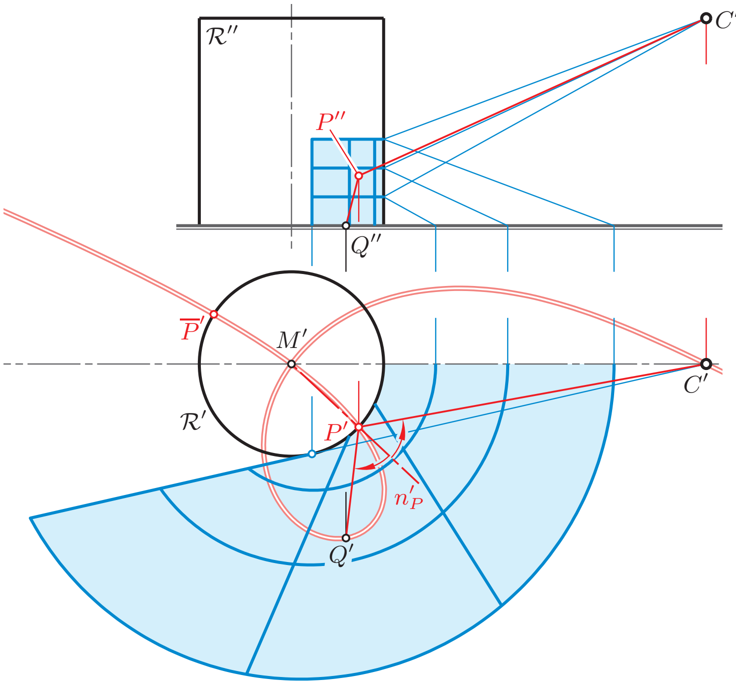

Figure 6: For a given center C and point Q in the ground plane the top views of the reflected images P and $\overline{P}$ lie on a strophoid

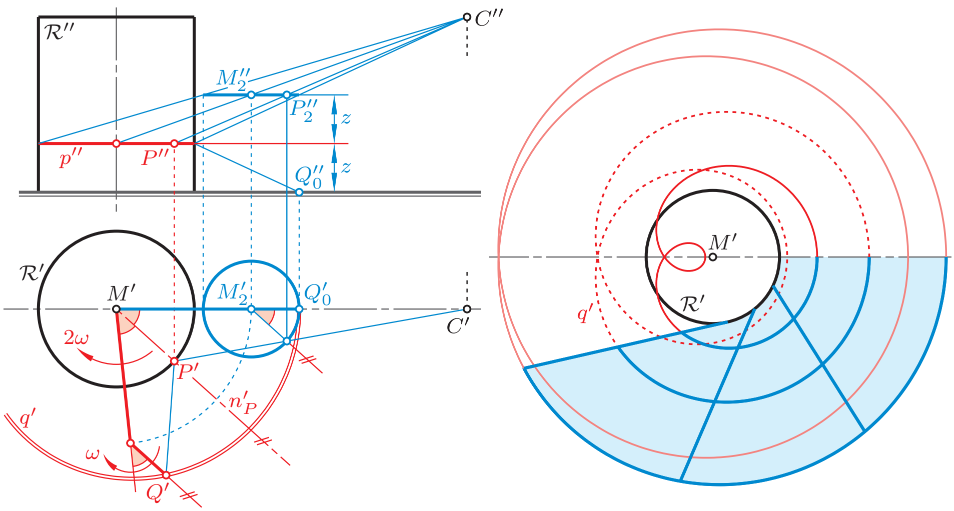

Figure 7: Parallel circles p on $\mathcal{R}$ are the reflected images of Pascal limaçons q in the ground plane

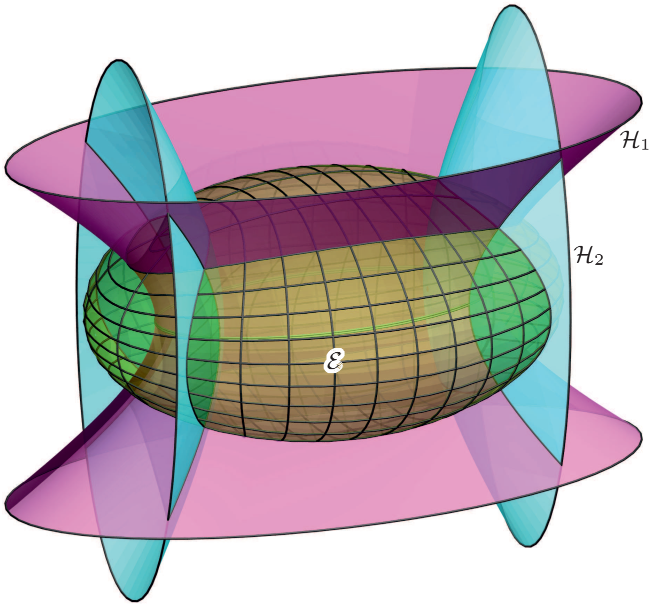

Figure 8: Confocal quadrics intersect mutually along their curvature lines (by courtesy of Boris Odehnal)

As limits for $k \to -c^2$ and $k \to -b^2$ we obtain 'flat' quadrics, i.e., the

$$\begin{aligned} &\textit{the focal ellipse} \quad f_e\colon \quad \frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1,\ z=0\,,\\ &\textit{the focal hyperbola} \quad f_h\colon \quad \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1,\ y=0\,. \end{aligned}$$

These two conics form a pair of focal conics: each is the locus of apices of right cones passing through the other conic [1, p. 137ff]. As a member of the confocal family, the two focal conics have to be seen as sets of tangent planes. Then they are rank 3 quadrics. According to this interpretation, all lines in space which meet any focal conic f in at least one point, are *tangent lines* of f. When below we speak of a *proper tangent line*, then we mean an ordinary tangent of the plane curve f.

The quadrics being confocal with an elliptic paraboloid $\mathcal{P}_e$ can be represented as

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}-2z-k=0 \quad \text{for} \quad k\in\mathbb{R}\setminus\{-a^2,-b^2\}. \tag{2.2}$$

In the case $a>b>0$ this one-parameter set includes

$$\text{for} \quad \begin{array}{ll} -b^2<k<\infty \quad \text{or} \quad k<-a^2 & \text{elliptic paraboloids } \mathcal{P}_e,\\ -a^2<k<-b^2 & \text{hyperbolic paraboloids } \mathcal{P}_h. \end{array}$$

For all k, the vertices of the paraboloids have the coordinates $(0,0,-k/2)$. Point $(0,0,b^2/2)$ is the common focal point of the principal sections in the plane $x=0$, and $(0,0,a^2/2)$ is the analogue for the sections with $y=0$.

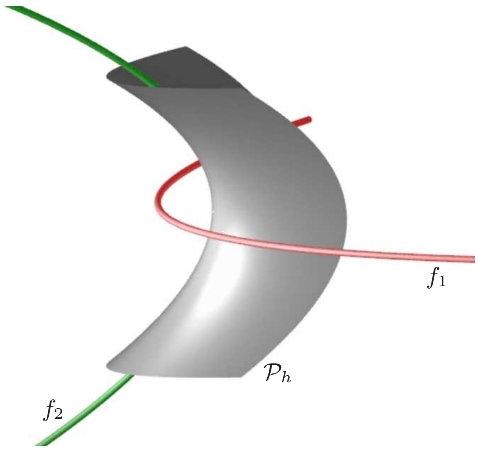

Figure 9: A hyperbolic paraboloid $\mathcal{P}_h$ together with its focal parabolas f_1 and f_2 (by courtesy of Georg Glaeser)

The limits for $k \to -a^2$ or $k \to -b^2$ define the pair of *focal parabolas*

$$\frac{y^2}{a^2-b^2} - 2z + b^2 = 0, \quad y = 0\,,$$
$$\frac{x^2}{a^2-b^2} + 2z + a^2 = 0, \quad x = 0$$

within the confocal family (Figure 9). For this pair of parabolas (compare with [1, Fig. 4.15] holds the same as mentioned above for an ellipse and its focal hyperbola.

For the sake of brevity, we ignore here the special cases of confocal quadrics of revolution. However, we recall that confocal quadratic cones can be given as

$$\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} - \frac{z^2}{c^2-k} = 0, \quad k \in \mathbb{R} \setminus \{-a^2, -b^2, c^2\}. \tag{2.3}$$

Their intersections with the unit sphere result in confocal spherical conics. If $a > b > 0$ then for $k \geq c^2$ and $k \leq -a^2$ the cones do not contain real points other than the origin. The 'flat' limit for $k \to -b^2$ is a sector bounded by the lines

$$\frac{x}{\sqrt{a^2-b^2}} \pm \frac{z}{\sqrt{b^2+c^2}} = 0 \tag{2.4}$$

in the plane $y=0$. These lines g_1, g_2 are called *focal lines* or *focal axes* of the cones, since they pass through the focal points of the corresponding spherical conics [1, p. 436ff]. The optical property, as shown in Figure 2, left, is also valid for spherical conics. Therefore the reflection in a quadratic cone transforms planes through one focal axis g_1 into planes through the other axis g_2.

In the case $a = b$ we obtain confocal cones of revolution. Their focal axes coincide in the common axis of revolution.

Theorem 2.2. *In dual setting, confocal quadrics form a one-parametric linear system (range) of quadrics sharing the isotropic tangent planes. Hence, the range includes the absolute conic as a rank-3 dual quadric.*
Similarily, confocal quadratic cones form a range, which includes the isotropic cone with the same apex. Since pairs of isotropic tangent planes of a quadratic cone intersect along a focal axis, confocal cones have common focal axes.

Proof. In order to obtain the tangential equations, we note that the plane satisfying

$$u_0 + u_1x + u_2y + u_3z = 0$$

is tangent to any surface of the confocal family (2.1) if and only if

$$(-u_0^2 + a^2u_1^2 + b^2u_2^2 + c^2u_3^2) + k(u_1^2 + u_2^2 + u_3^2) = 0.$$

This is a linear combination of the homogeneous dual equation of $\mathcal{E}$ and that of the set of isotropic planes. The homogeneous dual equations of confocal parabolas satisfying (2.2) have a similar form, namely

$$(a^2u_1^2 + b^2u_2^2 - 2u_0u_3) + k(u_1^2 + u_2^2 + u_3^2) = 0.$$

Finally, the dual equations of confocal cones, as given in (2.3), are

$$u_0 = 0, \quad (a^2u_1^2 + b^2u_2^2 - c^2u_3^2) + k(u_1^2 + u_2^2 + u_3^2) = 0,$$

and they show again a range, spanned by the given cone ($k = 0$) and the isotropic cone with their common apex at the origin. □

Theorem 2.3. *The cones or cylinders drawn from any finite or ideal point P tangent to the quadrics of a confocal family or connecting P with one of the included focal conics are confocal. For finite P, the common and mutually orthogonal planes of symmetry of these confocal cones are tangent to one of the three quadrics passing through P.*

Proof. The considered tangent cones share all isotropic planes which are common to the confocal quadrics and pass through P. Hence, the cones are confocal, too. This is a classical result attributed to C. G. J. Jacobi 1834 [8, p. 204] and a special case of a theorem concerning ranges of surfaces of degree 2.

The tangent cone from P to a quadric $\mathcal{Q}$ splits into pencils of planes with two real or complex conjugate axes if and only if $\mathcal{Q}$ passes through P. Then the two axes are generators of $\mathcal{Q}$ and span the tangent plane at P. On the other hand, the planes spanned by the axes of singular cones are the common planes of symmetry of the confocal cones. This confirms that confocal quadrics form a triply-orthogonal system of surfaces. □

Let a tangent line l of a quadric $\mathcal{Q}_0$ pass through any point P on the quadric $\mathcal{Q}$ being confocal with $\mathcal{Q}_0$. Then, by virtue of Theorem 2.3, the reflection of l at P in $\mathcal{Q}$ is again tangent to $\mathcal{Q}_0$, since the tangent plane τ_P to $\mathcal{Q}$ is a plane of symmetry of the cone of tangents drawn from P to $\mathcal{Q}_0$. Thus we obtain the spatial analogue of Theorem 1.1.

Corollary 2.4. *Let $\mathcal{Q}$ and $\mathcal{Q}_0$ be two different quadrics in a confocal family. Then the reflection in $\mathcal{Q}$ maps the line complex of tangents of $\mathcal{Q}_0$ onto itself. In particular, the complex of lines meeting any focal conic f of $\mathcal{Q}$ remains fixed.*

We only report that, in general, a given line contacts two surfaces of a confocal family, and the tangent planes at the respective points of contact are orthogonal (see, e.g., [9, p. 65]). This can be concluded from the spatial version of the Desargues involution theorem. However, there are exceptions, called *focal axes* [8, pp. 205–206]: Such a line l has the property that the isotropic planes through l are tangent to any quadric and therefore to all confocal quadrics.

Lemma 2.5. *Each focal axis l of a quadric $\mathcal{Q}$ is either a generator of a ruled quadric confocal with $\mathcal{Q}$ or a proper tangent of a focal conic of $\mathcal{Q}$. At each point $P \in l$, the focal axis l of $\mathcal{Q}$ is also a focal axis of the cone drawn from P tangent to $\mathcal{Q}$ or to any other confocal quadric.*

Proof. Each plane through a generator l of a ruled quadric is tangent to this quadric at a particular point of l. Therefore also the isotropic planes through l touch the quadric.
The tangent cone or cylinder with apex P comprises all tangent planes of $\mathcal{Q}$ which

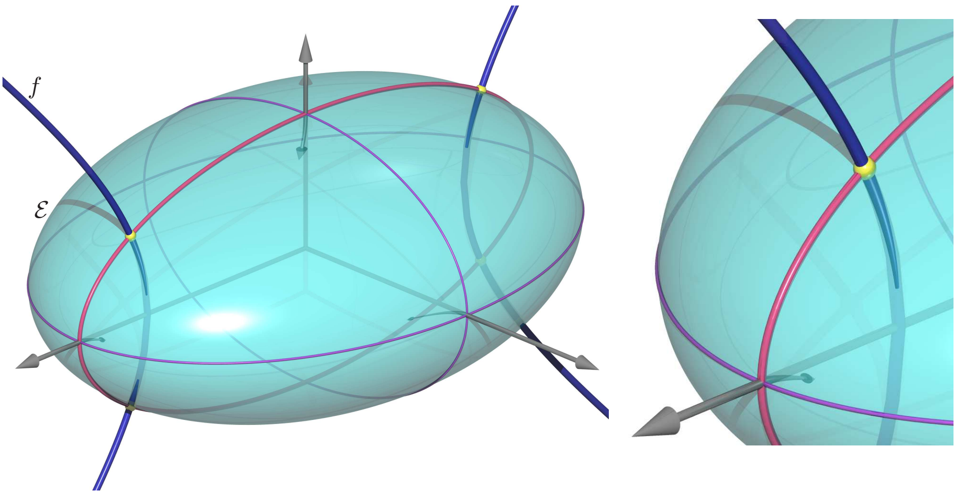

Figure 10: The perspective of the focal hyperbola coincides with its reflected image in the ellipsoid $\mathcal{E}$ (by courtesy of Boris Odehnal)

pass through P. If l is a focal axis of such a cone or cylinder then the isotropic planes through l are tangent to the cone and, hence, also to $\mathcal{Q}$. $\square$

Corollary 2.4 is the main reason for the optical effects mentioned at the beginning (Figure 1): Let a quadric $\mathcal{Q}$ and a central projection with center C be given. If any line l of sight, which meets a focal conic f of $\mathcal{Q}$ at a point Q_1, is reflected at the point $P \neq C$ in $\mathcal{Q}$, then the transformed line still meets f at any point Q_2. Hence, the perspective images of point Q_1 and P are coinciding, where P is the reflected image of Q_2 w.r.t. C. This holds for all $Q_1 \in f$. Therefore in the perspective the focal conic f and its reflected image in $\mathcal{Q}$ w.r.t. C belong to the same conic ("Theorem of the Transparent Cup").

The quadric in Figure 1 is a one-sheet hyperboloid of revolution, and f passes through the focal points of the meridians. In Figure 10 we have a reflecting ellipsoid $\mathcal{E}$ and its focal hyperbola f.

We can even replace the focal conic f by any other quadric in the confocal family and claim, as given below.

Corollary 2.6. *Let a reflecting quadric $\mathcal{Q}$ be given together with a confocal quadric $\mathcal{Q}_0$. Then in a perspective with any center C, the quadric $\mathcal{Q}_0$ and its reflected image in $\mathcal{Q}$ w.r.t. C have coinciding contours. This is also valid when $\mathcal{Q}_0$ degenerates into a focal conic f: The perspective of f coincides with that of its reflected image in $\mathcal{Q}$.*

3. Reflecting cones in a quadric

By virtue of Theorem 2.4, a line meeting a pair of focal conics f_1 and f_2 keeps this property after reflection in any quadric being confocal with f_1 and f_2. The set of such lines is the union of cones of revolution with apices on the focal conics. Now we check what happens if the generators of one of these cones are reflected.

Theorem 3.1. *Let $\mathcal{Q}$ be a quadric with focal conics f_1 and f_2. The cone $\mathcal{C}_0$ of revolution, which connects any point $S_0 \in f_1$ with f_2, intersects $\mathcal{Q}$ along two conics c_1 and c_2. The reflection in $\mathcal{Q}$ along the conic c_i, $i = 1, 2$, transforms $\mathcal{C}_0$ again in a cone $\mathcal{C}_i$ of revolution passing through f_2 with an apex $S_i \in f_1$ (Figure 11).*

Proof. The tangent cones drawn from point $S_0 \in f_1$ to the quadrics of the given confocal family are confocal with the cone $\mathcal{C}_0$ connecting S_0 with f_2. Since the latter one is a cone of revolution, they all are cones of revolution with the proper tangent t_{S_0} to f_1 at S_0 as their common axis. These cones are tangent to the isotropic planes through t_{S_0}; the respective lines of contact are isotropic lines in the plane orthogonal to t_{S_0} through S_0.
On the other hand, the poles of a fixed plane w.r.t. the quadrics of a range are collinear. For each isotropic plane through t_{S_0}, which touches all quadrics confocal with $\mathcal{Q}$, the points of contact are alined with two points: S_0 as the touching point with f_1, and the respective absolute point as the touching point with the absolute

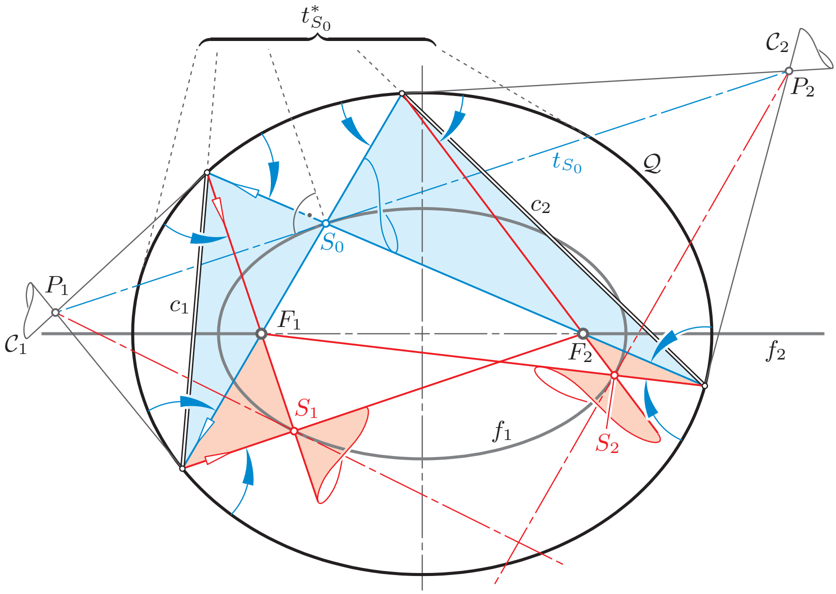

Figure 11: The reflection in the quadric $\mathcal{Q}$ transforms the right cone with apex $S_0 \in f_1$ onto two right cones with apices $S_1, S_2 \in f_1$

conic. Hence, the quadric $\mathcal{Q}$, like any other confocal quadric, contacts the cone $\mathcal{C}_0$ at two points. Consequently, the curve of intersection $\mathcal{Q} \cap \mathcal{C}_0$ splits into two conics c_1 and c_2, both passing through the points of contact on the line $t^*_{S_0}$, polar to t_{S_0} w.r.t. $\mathcal{Q}$. Figure 11 shows the scene after being orthogonally projected into the plane of the focal conic f_1.

Let P_i denote the apex of the tangent cone $\mathcal{C}_i$ of $\mathcal{Q}$ along c_i for $i = 1, 2$. In accordance with Lemma 2.5, the two proper tangents drawn from P_i to f_1 are the focal axes of $\mathcal{C}_i$. One of them is t_{S_0}, the other contacts f_1 at S_i (Figure 11). As already noted, the reflection in $\mathcal{C}_i$ transforms planes through t_{S_0} into planes through P_iS_i. Due to the contact between $\mathcal{C}_i$ of $\mathcal{Q}$ along c_i, for each point $X \in c_i$ the reflection in $\mathcal{Q}$ maps the line S_0X onto a line meeting the axis P_iS_i. On the other hand, by virtue of Corollary 2.4, the reflected line must also meet f_1 (and f_2). Hence, the reflection of S_0X coincides with S_iX, as stated in Theorem 3.1. For all $X \in c_i$, the planes spanned by the incoming and outgoing ray, which contain also the surface normal n_X to $\mathcal{Q}$, have the common trace S_0S_i in the plane of f_1. $\square$

The given proof reveals that Theorem 3.1 can be generalized by replacing the focal conic f_2 with any confocal quadric $\mathcal{Q}_0$.

Theorem 3.2. *Let $\mathcal{Q}$ and $\mathcal{Q}_0$ be two confocal quadrics. Then the reflection in $\mathcal{Q}$ transforms each cone of revolution, which is tangent to $\mathcal{Q}_0$, into two cones of the same type.*

Remark 3.3. It can be shown that, conversely, the only smooth cones which by reflection in a general quadric correspond again to a cone, are those mentioned in Theorem 3.2.

From a limiting case of Theorem 3.1 we learn how the well known reflecting property of a satellite-TV receiving dish changes when the paraboloid of revolution is replaced with a general elliptic paraboloid.

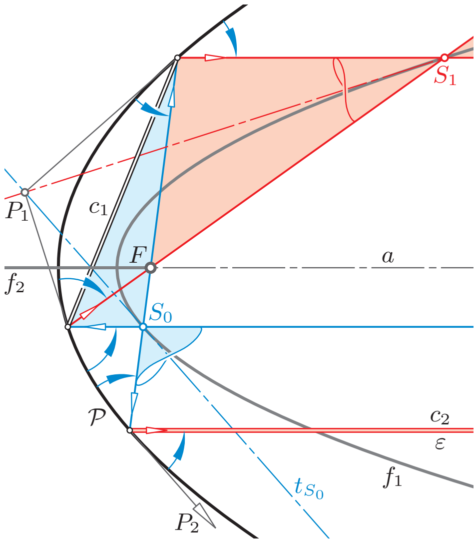

Figure 12: The reflection in the elliptic paraboloid $\mathcal{P}$ transforms the right cone with apex $S_0 \in f_1$ onto the right cone with apex $S_1 \in f_1$ and a pencil of lines parallel to the axis a in the plane ε

Theorem 3.4. *Let $\mathcal{P}$ be any paraboloid other than a paraboloid of revolution. Then the reflection in $\mathcal{P}$ maps all lines l being parallel to the axis a of $\mathcal{P}$ onto lines meeting both focal parabolas f_1 and f_2 of $\mathcal{P}$. The pencil of those parallels l to a, which lie in a plane ε orthogonal to the plane of f_1, is mapped onto a cone of revolution with apex $S_0 \in f_1$.*

The latter can also be concluded as follows (see Figure 12). Let c_2 denote the parabola $\mathcal{P} \cap \varepsilon$. The tangent cone of $\mathcal{P}$ along c_2 is a parabolic cylinder $\mathcal{C}_2$ with apex P_2 at infinity. After an orthogonal projection with center P_2 the cylinder $\mathcal{C}_2$ appears as a parabola $\mathcal{C}_2^n$. In this view the reflection in $\mathcal{Q}$ along c_2 is seen as a planar reflection in $\mathcal{C}_2^n$ which transforms lines parallel to the parabola's axis onto lines through the focus of $\mathcal{C}_2^n$. This focus coincides with the view of S_0, which is the point of f_1 with the proper tangent t_{S_0} passing through P_2.

Remark 3.5. The bundle of parallels to the axis a of the paraboloid $\mathcal{P}$ consists of all lines orthogonal to a plane. By virtue of the Theorem of Malus and Dupin [6, p. 446], the property of being a *normal line congruence* is preserved under reflection in a surface. The surfaces orthogonal to the lines meeting the pair of focal parabolas of $\mathcal{P}$ are parabolic Dupin cyclides [1, p. 147ff]. We recall that the surfaces, whose normals intersect an ellipse and its focal hyperbola, are general Dupin cyclides.

References

[1] Glaeser, G., Stachel, H., Odehnal, B., *The Universe of Conics*, Springer Spectrum, Berlin Heidelberg 2016.

[2] Glaeser, G., Odehnal, B., Stachel, H., *The Universe of Quadrics*, Springer Spectrum (in preparation).

[3] Izmestiev, I., Tabachnikov, S., *Ivory's Theorem revisited*, Journal of Integrable Systems 2/1, xyx006 (2017) (`https://doi.org/10.1093/integr/xyx006`).

[4] Knöbl, M., *The Transparent Cup Theorem*, Retrieved from [`http://karuga.eu/transparent-cup.html`], 2016.

[5] Mazzalai, S., *Between Memory and Innovation: Algorithmic Analysis of some Catoptric Anamorphoses by Jean François Niceron,* Proceedings of the 17th ICGG, Beijing 2016, no. 27.

[6] Pottmann, H., Wallner, J., *Computational Line Geometry,* Springer-Verlag, Berlin 2001.

[7] Stachel, H., *Strophoids are auto-isogonal cubics.* G – Slovak Journal for Geometry and Graphics, ISSN 1336-524X, **12**, no. 24, 45–59 (2015).

[8] Staude, O.J., *Flächen 2. Ordnung und ihre Systeme und Durchdringungskurven*, in *Encyklopädie der math. Wiss.* III.2.1, no. C2, 161–256, B.G. Teubner, Leipzig 1915.

[9] Tabachnikov, S., *Geometry and Billiards,* American Mathematical Society, Providence/Rhode Island 2005.

[10] Wunderlich, W., *Ebene Kinematik,* Bibliographisches Institut, Mannheim 1970.